Acta Sci. Math., 56 (1992), 147—163

Shorts of operators and some extremal problems

E. L. PEKAREV

0. Introduction

Let $\mathscr{H}$ be a Hilbert space, $\mathscr{L}$ its closed subspace, and A a non-negative operator in $\mathscr{H}$. As M. G. KREIN [1] showed, the set of operators

$$\mathscr{X}(A, \mathscr{L}) = \{X\colon 0 \leqq X \leqq A, \mathscr{R}(X) \subset \mathscr{L}\} \tag{0.1}$$

contains a maximum element, denoted by $A_{\mathscr{L}}$ and called the *short* of A to $\mathscr{L}$:

$$A_{\mathscr{L}} = \max \mathscr{X}(A, \mathscr{L}). \tag{0.2}$$

The properties of the correspondence $A \mapsto A_{\mathscr{L}}$ were studied in detail in [2] and found various applications to the theory of characteristic operator-functions [3, 4], electrical networks [5, 6], Lebesgue decomposition of nonnegative operators and positive definite operator-functions [7—9], operator means [10] and other problems. The notion of short was generalized to the case of non-closed $\mathscr{L}$, which is the range of a bounded operator, and it became clear that shorting is closely connected with the operation of parallel addition arising in the theory of electrical networks and with its inverse operation, parallel subtraction.

A recent work of S. L. ERIKSSON and LEUTWILLER [13], related to parallel addition, indicates essential connection between shorts and extreme points of some set of operators. In the present note we continue the study of this connection, and also give proofs to some assertions, announced earlier in [12, 14].

For Hilbert spaces $\mathscr{G}$, $\mathscr{H}$, let us denote by $\mathscr{B}(\mathscr{G}, \mathscr{H})$ the class of all bounded linear operators from $\mathscr{G}$ to $\mathscr{H}$. For $\mathscr{L} \subset \mathscr{H}$ and $T \in \mathscr{B}(\mathscr{G}, \mathscr{H})$, we denote by $T^{-1}\mathscr{L} (\subset \mathscr{G})$ the preimage of $\mathscr{L}$ under T, $\mathscr{N}(T) = T^{-1}\{0\}$ and $\mathscr{R}(T) = T\mathscr{G}$. When $\mathscr{G} = \mathscr{H}$, we shall write $\mathscr{B}(\mathscr{H}) = \mathscr{B}(\mathscr{H}, \mathscr{H})$. The class of all non-negative operators in $\mathscr{B}(\mathscr{H})$ is denoted by $\mathscr{B}_+(\mathscr{H})$.

Received March 7, 1990.

1. Shorts and complements

Let $\mathscr{G}$, $\mathscr{H}$ be Hilbert spaces, $\mathscr{F}=\mathscr{G}\oplus\mathscr{H}$, T an operator $\in\mathscr{B}(\mathscr{G},\mathscr{H})$. In accordance with [11] let

$$\{\mathscr{H}\}_T = \{A\in\mathscr{B}_+(\mathscr{H})\colon \mathscr{R}(A^{1/2}) \supset \mathscr{R}(T)\}.$$

The class $\{\mathscr{H}\}_T$ consists of operators $A\in\mathscr{B}_+(\mathscr{H})$ such that $\begin{bmatrix} G & T^* \\ T & A \end{bmatrix} \geqq 0$ for some $G\in\mathscr{B}_+(\mathscr{G})$. If $A\in\{\mathscr{H}\}_T$, then among those $G\in\mathscr{B}_+(\mathscr{G})$ there is a minimum, called the *complement* of A relative to T denoted by A_T. Namely, $A_T=WW^*$, where the operator $W\in\mathscr{B}(\mathscr{H},\mathscr{G})$ is uniquely determined by the condition

$$WA^{1/2} = T^*, \quad \mathscr{N}(W) \supset \mathscr{N}(A). \tag{1.1}$$

If $A\in\mathscr{B}_+(\mathscr{H})$ is invertible, then $A\in\{\mathscr{H}\}_T$ for any $T\in\mathscr{B}(\mathscr{G},\mathscr{H})$ and $A_T=$ $=T^*A^{-1}T$. In the general case by [2] the following identity is fulfilled

$$(A_Tg, g) = \sup_{h\in\mathscr{H}}\{(Tg, h)+(T^*h, g)-(Ah, h)\} = \sup_{h\in\mathscr{H}} \frac{|(h, Tg)|^2}{(Ah, h)}. \tag{1.2}$$

The class $\{\mathscr{G}\}_{T^*}$ and the complement $G_{T^*}\in\mathscr{B}_+(\mathscr{H})$ are defined correspondingly. The mappings $A\mapsto A_T$ and $G\mapsto G_{T^*}$ define a Galois correspondence between the classes $\{\mathscr{H}\}_T$ and $\{\mathscr{G}\}_{T^*}$ (if in these classes the order relations are defined inversely to the usual ones) [11]. This gives rise to a closure operations $A\mapsto(A_T)_{T^*}\equiv\gamma_T(A)$ and $G\mapsto(G_{T^*})_T\equiv\gamma_{T^*}(G)$ in $\{\mathscr{H}\}_T$ and $\{\mathscr{G}\}_{T^*}$ respectively. These operations are monotone in their respective classes. Thus if, for instance, A_1 and $A_2\in\{\mathscr{H}\}_T$ and $A_1\leqq A_2$, then $\gamma_T(A_1)\leqq\gamma_T(A_2)$. Besides, for any $A\in\{\mathscr{H}\}_T$ ($G\in\{\mathscr{G}\}_{T^*}$) we have $\gamma_T(A)\leqq A\big(\gamma_{T^*}(G)\leqq G\big)$. In case $A=\gamma_T(A)$ $\big(G=\gamma_{T^*}(G)\big)$, then the operator $A(G)$ is called *T-closed (T^*-closed)*. The class of T-closed (T^*-closed) operators is denoted by $[\mathscr{H}]_T$ ($[\mathscr{G}]_{T^*}$).

The proof of the following theorem is found in [11].

Theorem 1.1. *In order that an operator $A\in\{\mathscr{H}\}_T$ belongs to $[\mathscr{H}]_T$ it is necessary and sufficient that the equality $\overline{A^{-1/2}\mathscr{R}(T)}=\mathscr{H}$ holds.*

Theorem 1.2. *If $\mathscr{R}(T_1)=\mathscr{R}(T_2)$, then $\{\mathscr{H}\}_{T_1}=\{\mathscr{H}\}_{T_2}$, $[\mathscr{H}]_{T_1}=[\mathscr{H}]_{T_2}$ and $\gamma_{T_1}(A)=\gamma_{T_2}(A)$ for any $A\in\{\mathscr{H}\}_{T_1}$.*

Proof. Since for any operator T

$$\{\mathscr{H}\}_T = \{A\in\mathscr{B}_+(\mathscr{H})\colon \mathscr{R}(A^{1/2}) \supset \mathscr{R}(T)\},$$

$\{\mathscr{H}\}_{T_1}=\{\mathscr{H}\}_{T_2}$ whenever $\mathscr{R}(T_1)=\mathscr{R}(T_2)$. Now it follows easily from the preceding proposition that $[\mathscr{H}]_{T_1}=[\mathscr{H}]_{T_2}$. Then $\gamma_{T_2}(\gamma_{T_1}(A))=\gamma_{T_1}(A)$ for any $A\in\{\mathscr{H}\}_{T_1}$, hence $\gamma_{T_2}(A)\leqq A$ implies $\gamma_{T_2}(A)=\gamma_{T_1}(A)$. Exchanging the roles of T_1 and T_2, we have $\gamma_{T_2}(A) \geqq \gamma_{T_1}(A)$. Thus for any $A \in \{\mathcal{H}\}_{T_1}$ the expected equality $\gamma_{T_1}(A) = \gamma_{T_2}(A)$ is satisfied.

Corollary. *If $\mathcal{G} = \mathcal{H}$, $T \in \mathcal{B}(\mathcal{H})$ and $|T^*| = (TT^*)^{1/2}$, then $\{\mathcal{H}\}_T = \{\mathcal{H}\}_{|T^*|}$ and $\gamma_T(A) = \gamma_{|T^*|}(A)$ for any $A \in \{\mathcal{H}\}_T$.*

In many occasions it is convenient to use the following formula, established in [11].

(1.3)
$$\gamma_T(A) = A^{1/2} P_{\mathcal{M}} A^{1/2} \quad (A \in \{\mathcal{H}\}_T),$$

where $P_{\mathcal{M}}$ is the orthogonal projection to the subspace $\mathcal{M} = \overline{A^{-1/2}\mathcal{R}(T)}$.

Suppose that there is given an *operator range* $\mathcal{L}$, that is, the range of a bounded operator. In view of Theorem 1.2 it is possible to define the classes $\{\mathcal{H}\}_{\mathcal{L}}$, $[\mathcal{H}]_{\mathcal{L}}$ and the operation $\gamma_{\mathcal{L}}(A)$. More precisely, if $\mathcal{L} = \mathcal{R}(T)$ for $T \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ with some $\mathcal{G}$, then let $\{\mathcal{H}\}_{\mathcal{L}} = \{\mathcal{H}\}_T$, $[\mathcal{H}]_{\mathcal{L}} = [\mathcal{H}]_T$, $\gamma_{\mathcal{L}}(A) = \gamma_T(A)$ for $A \in \{\mathcal{H}\}_T$.

Formula (1.3) is written in the form

(1.4)
$$\gamma_{\mathcal{L}}(A) = A^{1/2} P_{\mathcal{M}} A^{1/2} \quad \text{for} \quad A \in \{\mathcal{H}\}_{\mathcal{L}},$$

where $P_{\mathcal{M}}$ is the orthogonal projection to the subspace $\mathcal{M} = \overline{A^{-1/2}\mathcal{L}}$.

Remark that an operator range $\mathcal{L}_1$ is contained in $\mathcal{L}$, $\mathcal{L}_1 \subset \mathcal{L} (\subset \mathcal{H})$, then $\{\mathcal{H}\}_{\mathcal{L}} \subset \{\mathcal{H}\}_{\mathcal{L}_1}$ and for $A \in \{\mathcal{H}\}_{\mathcal{L}}$, by (1, 4),

(1.5)
$$\gamma_{\mathcal{L}_1}(A) \leqq \gamma_{\mathcal{L}}(A).$$

Theorem 1.3. *Let $\mathcal{L} \subset \mathcal{H}$ be an operator range, and $A \in \{\mathcal{H}\}_{\mathcal{L}}$. Then $\gamma_{\mathcal{L}}(A)$ is a maximum element of the set*

(1.6)
$$\mathcal{X}_0(A, \mathcal{L}) = \{X\colon\ 0 \leqq X \leqq A, \overline{X^{-1/2}\mathcal{L}} = \mathcal{H}\}.$$

In particular, if $\overline{\mathcal{L}} = \mathcal{L}$, then $\gamma_{\mathcal{L}}(A)$ is also a maximum element in the set (0.1).

Proof. Remark that if $X \in \mathcal{X}_0(A, \mathcal{L})$ then $X \in \mathcal{X}_0(A, \mathcal{L}_1)$ too, where $\mathcal{L}_1 = \mathcal{R}(X^{1/2}) \cap \mathcal{L}$ is also an operator range (see [15]). Here $X \in \{\mathcal{H}\}_{\mathcal{L}_1}$ and further more $X \in [\mathcal{H}]_{\mathcal{L}_1}$, because $\overline{X^{-1/2}\mathcal{L}_1} = \mathcal{H}$. Therefore $X = \gamma_{\mathcal{L}_1}(X) \leqq \gamma_{\mathcal{L}_1}(A)$ because of the monotoneness of $\gamma_{\mathcal{L}_1}(\cdot)$, and it follows from (1.5) that $X \leqq \gamma_{\mathcal{L}}(A)$. Together with $\gamma_{\mathcal{L}}(A) (\in [\mathcal{H}_{\mathcal{L}}])$, by Theorem 1.1, $\mathcal{X}_0(A, \mathcal{L})$ contains

$$\gamma_{\mathcal{L}}(A) = \max \mathcal{X}_0(A, \mathcal{L}).$$

Suppose, in particular, that $\overline{\mathcal{L}} = \mathcal{L}$. Then we have by (1.4)

$$\mathcal{R}(\gamma_{\mathcal{L}}(A)) \subset \mathcal{R}(\gamma_{\mathcal{L}}(A)^{1/2}) = \mathcal{B}(A^{1/2} P_{\mathcal{M}}) \subset \mathcal{L},$$

hence $\gamma_{\mathcal{L}}(A) \in \mathcal{X}(A, \mathcal{L})$. Since obviously $\mathcal{X}(A, \mathcal{L}) \subset \mathcal{X}_0(A, \mathcal{L})$, $\gamma_{\mathcal{L}}(A)$ is a maximum operator in $\mathcal{X}(A, \mathcal{L})$, what is to prove.

Now consider an arbitrary operator range $\mathscr{L}\subset\mathscr{H}$ and an arbitrary operator $A\in\mathscr{B}(\mathscr{H})$ (not supposing $A\in\{\mathscr{H}\}_{\mathscr{L}}$). With $\mathscr{L}_1=\mathscr{L}\cap\mathscr{R}(A^{1/2})$, we have $A\in\{\mathscr{H}\}_{\mathscr{L}_1}$ and $\mathscr{X}_0(A,\mathscr{L})=\mathscr{X}_0(A,\mathscr{L}_1)$. Therefore according to Theorem 1.3 the set $\mathscr{X}_0(A,\mathscr{L})$ has a maximum element $\max\mathscr{X}_0(A,\mathscr{L})=\gamma_{\mathscr{L}_1}(A)$. As in Theorem 1.3 it is not difficult to see that if $\overline{\mathscr{L}}=\mathscr{L}$ then $\max\mathscr{X}_0(A,\mathscr{L})=\max\mathscr{X}(A,\mathscr{L})$.

In accordance with (0.2) let us introduce

Definition [12]. The *short* of an operator $A\in\mathscr{B}_+(\mathscr{H})$ to an operator range $\mathscr{L}\subset\mathscr{H}$ is the operator, defined by the relation

$$A_{\mathscr{L}} = \max\mathscr{X}_0(A,\mathscr{L}). \tag{1.7}$$

Since $A_{\mathscr{L}}=\gamma_{\mathscr{L}_1}(A)$ with $\mathscr{L}_1=\mathscr{L}\cap\mathscr{R}(A^{1/2})$, by (1.4) we have immediatley the following representation:

Theorem 1.4. *If $A\in\mathscr{B}_+(\mathscr{H})$ and $\mathscr{L}$ is an operator range $\subset\mathscr{H}$, then*

$$A_{\mathscr{L}} = A^{1/2}P_{\mathscr{M}}A^{1/2} \tag{1.8}$$

where $P_{\mathscr{M}}$ is the orthogonal projection to the subspace $\mathscr{M}=\overline{A^{-1/2}\mathscr{L}}$.

Corollary 1. $(A^2)_{\mathscr{L}}\leqq(A_{\mathscr{L}})^2$.

In fact, let $\mathscr{M}=\overline{A^{-1/2}\mathscr{L}}$, $\mathscr{M}_1=\overline{A^{-1}\mathscr{L}}$, and $P_{\mathscr{M}}$ and $P_{\mathscr{M}_1}$ the orthogonal projections to $\mathscr{M}$ and $\mathscr{M}_1$, respectively. If $g\in A^{-1}\mathscr{L}$ then $Ag\in\mathscr{L}$, and $A^{1/2}g\in A^{-1/2}\mathscr{L}\subset\mathscr{M}$. Therefore $P_{\mathscr{M}_1}A^{1/2}(I-P_{\mathscr{M}})=0$ and for any $h\in\mathscr{H}$ we have the inequality $\|P_{\mathscr{M}_1}A^{1/2}h\|=$ $=\|P_{\mathscr{M}_1}A^{1/2}(I-P_{\mathscr{M}})h\|+\|P_{\mathscr{M}_1}A^{1/2}P_{\mathscr{M}}h\|\leqq\|A^{1/2}P_{\mathscr{M}}h\|$. This implies that $A^{1/2}P_{\mathscr{M}_1}A^{1/2}\leqq$ $\leqq P_{\mathscr{M}}AP_{\mathscr{M}}$, and hence $AP_{\mathscr{M}_1}A\leqq A^{1/2}P_{\mathscr{M}}AP_{\mathscr{M}}A^{1/2}$. The last inequality means, by Theorem 1.4, that $(A^2)_{\mathscr{L}}\leqq(A_{\mathscr{L}})^2$.

Corollary 2. $A_{\mathscr{L}}=A_{\mathscr{L}_1}$ *if and only if* $\overline{A^{-1/2}\mathscr{L}}=\overline{A^{-1/2}\mathscr{L}_1}$. *In particular,* $A_{\mathscr{L}}=A_{\overline{\mathscr{L}}}$ *if and only if* $\overline{A^{-1/2}\mathscr{L}}=\overline{A^{-1/2}\overline{\mathscr{L}}}$.

It is easy to construct an example in which the last equality does not take place In fact, if $\mathscr{R}(A)\neq\overline{\mathscr{R}(A)}=\mathscr{H}$ and $h_0\notin\mathscr{R}(A^{1/2})$, let $\mathscr{L}=\{A^{1/2}h\colon h\in\mathscr{H}, h\perp h_0\}$ to get $h_0\perp\overline{A^{-1/2}\mathscr{L}}$. Then as $\overline{\mathscr{L}}=\mathscr{H}$, $A^{-1/2}\overline{\mathscr{L}}=\mathscr{H}$.

Clearly if $\mathscr{L}=\overline{\mathscr{L}}(\subset\mathscr{H})$, $A\in\mathscr{B}_+(\mathscr{H})$ and $\mathscr{L}'=\mathscr{H}\ominus\mathscr{L}$, then with respect to the orthogonal decomposition $\mathscr{H}=\mathscr{L}\oplus\mathscr{L}'$, we have a matrix representation of operators

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}, \quad A_{\mathscr{L}} = \begin{bmatrix} A_0 & 0 \\ 0 & 0\end{bmatrix}$$

where $A_0=A_{11}-(A_{22})_{A_{12}}$, hence by (1.2)

$$(A_{\mathscr{L}}h,h) = \inf_{g\perp\mathscr{L}}(A(h+g),h+g) \quad (h\in\mathscr{H}). \tag{1.9}$$

In concluding this section, remark that in case $\mathscr{L}=\overline{\mathscr{L}}$. the operator $A_{\mathscr{L}}(=\max \mathscr{X}(A, \mathscr{L})=\max \mathscr{X}_0(A, \mathscr{L}))$ becomes, obviously, the maximum in the set

$$\mathscr{X}'(A, \mathscr{L}) = \{X\colon 0 \leqq X \leqq A, \mathscr{R}(X^{1/2}) \subset \mathscr{L}\} \tag{1.10}$$

(because $\mathscr{X}'(A, \mathscr{L})=\mathscr{X}(A, \mathscr{L})$ in case $\mathscr{L}=\overline{\mathscr{L}}$.) If $\mathscr{L}\neq\overline{\mathscr{L}}$, as shown in the following section, the sets (0.1) and (1.10) do not contain maximum elements in general.

2. Short, parallel addition and parallel subtraction

In the theory of electrical network the parallel sum of invertible operators (matrices) A and B corresponding to the impedances of branches of the network, is the operator $A\colon B=(A^{-1}+B^{-1})^{-1}$, which becomes the impedance of the parallel connection of those branches. When A and B are non-negative, their parallel sum is suitably defined [16, 5, 11]*).

Let $\mathscr{H}$ be a Hilbert space, and $A, B\in\mathscr{B}_+(\mathscr{H})$. The operator

$$\mathbf{A} = \begin{bmatrix} A & A \\ A & A+B \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} I & I \\ 0 & I \end{bmatrix}, \tag{2.1}$$

acting on the space $\mathscr{F}=\mathscr{H}\oplus\mathscr{H}$, belongs to $\mathscr{B}_+(\mathscr{F})$, so that $A+B\in\{\mathscr{H}\}_{\mathbf{A}}$ and $A\geqq(A+B)_{\mathbf{A}}$.

Definition [5, 11]. The operator

$$A\colon B = A-(A+B)_{\mathbf{A}} \quad (\geqq 0) \tag{2.2}$$

is called the *parallel sum* of A and B.

It is clear that if A and B are invertible then

$$A\colon B = A-A(A+B)^{-1}A = A(A+B)^{-1}B = (A^{-1}+B^{-1})^{-1}.$$

Since the short of the operator $\mathbf{A}$ in (2.1) to the first component $\mathscr{H}$ coincides with $\begin{bmatrix} A\colon B & 0 \\ 0 & 0 \end{bmatrix}$, we have from (1.9)

$$((A\colon B)f, f) = \inf_{\substack{g, h\in\mathscr{H} \\ g+h=f}} \{(Ag, g)+(Bh, h)\}. \tag{2.3}$$

*) Remark that in a number of papers the operation of parallel addition is extended to a wider class of operators, in particular, to a class of non-linear operators [17, 18, 19].

The following properties of parallel sum follow easily from (2.2) and (2.3) (see, for instance [5, 6], and also [11]):

$$A:B = B:A, \quad (A:B):C = A:(B:C) \tag{2.4}$$

$$C(A:B)C \leqq (CAC):(CBC) \tag{2.5}$$

$$(A+B):(C+D) \geqq A:C+B:C \tag{2.6}$$

$$A_n \downarrow A, \quad B_n \downarrow B \Rightarrow A_n:B_n \downarrow A:B \tag{2.7}$$

$$\mathscr{R}\big((A:B)^{1/2}\big) = \mathscr{R}(A^{1/2}) \cap \mathscr{R}(B^{1/2}). \tag{2.8}$$

All operators in (2.4)—(2.8) are assumed non-negative. Notice that without any additional condition in (2.5) equality sign may not occur even in the two dimensional case.

Example. Let $A=\begin{bmatrix}1 & \alpha\\ \alpha & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & \beta\\ \beta & 1\end{bmatrix}$ where $0<\alpha, \beta<1$, $\alpha\neq\beta$ and $C=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$. Then equality sign does not occur in (2.5), because

$$(CAC):(CBC) = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad C(A:B)C = \frac{1}{\Delta}\begin{pmatrix}2-(\alpha^2+\beta^2) & 0\\ 0 & 0\end{pmatrix}$$

where $\Delta = \det(A+B) = 4-(\alpha+\beta)^2$.

Lemma 1. *If* $A, B, C \in \mathscr{B}_+(\mathscr{H})$, *and* $\overline{\mathscr{R}(A)}+\overline{\mathscr{R}(B)} \subset \overline{\mathscr{R}(C)}$, *then*

$$C(A:B)C = (CAC):(CBC).$$

Proof. With $\mathscr{L}=\mathscr{R}(C)$ and $h'=Ch$, using formula (2.3) we have

$$\begin{aligned}((CAC):(CBC)f,f) &= \inf_{h\in\mathscr{H}}\{(AC(f-h), C(f-h))+(BCh, Ch)\} = \\ &= \inf_{h'\in\mathscr{L}}\{(A(Cf-h'), Cf-h')+(Bh', h')\}.\end{aligned}$$

The infimum does not change even if $\mathscr{L}$ is replaced by $\overline{\mathscr{L}}$. On the other hand, putting $g=h'+h''$, where h' runs over $\overline{\mathscr{L}}$ and h'' over $\mathscr{H}\ominus\mathscr{L}$, since $Ah''=Bh''=0$, we have

$$\begin{aligned}(C(A:B)Cf,f) &= \inf_{g\in\mathscr{H}}\{(A(Cf-g), Cf-g)+(Bg, g)\} = \\ &= \inf_{h'\in\overline{\mathscr{L}}}\{(A(Cf-h'), Cf-h')+(Bh', h')\}.\end{aligned}$$

This completes the proof.

Remark that all operators in (2.7) are invertible then convergence from above can be changed to convergence from below, though this is not the case in general. At the same time parallel sum becomes a non-decreasing function of "summands", and $0\leqq A:B\leqq A, B$. Hence for any non-decreasing sequences $\{A_n\}$, $\{B_n\}\subset\mathscr{B}_+(\mathscr{H})$ the sequence $\{A_n:B_n\}$ is also non-decreasing, and if one of the given sequences is bounded, then a limit $\lim_{n\to 0} A_n:B_n$ exists. Supposing that A_n, B_n $(n=1, 2, \ldots)$ are invertible, let $A_\infty=\lim_{n\to\infty} A_n^{-1}$, $B_\infty=\lim_{n\to\infty} B_n^{-1}$. Then it is clear that

$$(A_n+B_n)^{-1} = A_n^{-1}:B_n^{-1}\downarrow A_\infty:B_\infty .$$

Let the sequence $\{A_n\}$ be bounded and $A=\lim_{n\to\infty} A_n(=A_\infty^{-1})$. Then

$$A_n:B_n = A_n-A_n(A_n+B_n)^{-1}A_n \xrightarrow[n\to\infty]{} A-A(A_\infty:B_\infty)A$$

and by Lemma 2.1

$$\lim_{n\to\infty} A_n:B_n = A-A:(AB_\infty A).$$

It follows, in particular, that with $A_n=A$ and $B_n=nB$

$$\lim_{n\to\infty} A:nB=A$$

for any invertible $B\in\mathscr{B}_+(\mathscr{H})$. Here the last equality is obvioualy satisfied for any $A\in\mathscr{B}_+(\mathscr{H})$, because

$$A:nB = A-A(A+nB)^{-1}A \quad \text{and} \quad (A+nB)^{-1}\downarrow 0.$$

For a non-invertible operator $B\in\mathscr{B}_+(\mathscr{H})$ the following assertion is valid (see [12, 21]):

Theorem 2.2. *If $A, B\in\mathscr{B}_+(\mathscr{H})$, then $C=\lim_{n\to\infty} A:nB$ is a minimum non-negative solution of the equation $X:B=A:B$.*

Recall that if B, $S\in\mathscr{B}_+(\mathscr{H})$, then the equation $X:B=S$ with unknown $X\in\mathscr{B}_+(\mathscr{H})$ has a solution exactly when $B-S\in[\mathscr{H}]_B$, and in this case there exists a minimum solution.

Definition [11]. The minimum solution of the equation $X:B=S$ is called the *parallel difference* of operators S and B, and is denoted by $S\div B$.

We can prove that

$$S\div B = (B-S)_B-B = (B-S)_S+S, \tag{2.9}$$

hence by (1.2) we have

$$((S\div B)f,f) = \sup_{g\in\mathscr{H}}\{(S(f+g),f+g)-(Bg,g)\} \quad (\forall f\in\mathscr{H}). \tag{2.10}$$

It is clear that if operator $B-S$ is continuously invertible then Theorem 1.1 implies that the parallel difference $S\div B$ exists, and by (2.9)

$$S\div B = B(B-S)^{-1}B-B = B(B-S)^{-1}S.$$

Let us cite, for completeness, some properties of parallel subtraction, proved in [11] (provided that parallel subtraction in question exists):

$$S_1 \geqq S_2, \quad B_1 \leqq B_2 \Rightarrow S_1 \div B_1 \geqq S_2 \div B_2$$
$$S = A:B \Rightarrow S \div (S \div B) = S \div A$$
$$S = A:B \Rightarrow (S \div B):(S \div A) = S$$
$$S_n \uparrow S, \quad A_n \downarrow A \Rightarrow S_n \div A_n \uparrow S \div A$$

Given $S \in \mathscr{B}_+(\mathscr{H})$, let $\mathscr{M}(S)$ denote the class of those $A \in \mathscr{B}_+(\mathscr{H})$, for which the equation $A:X=S$ is solvable. When twice applied, the mapping $A \mapsto S \div A$ defines a Galois correspondence between $\mathscr{M}(S)$ and $\mathscr{M}(S)$: $A \mapsto S \div (S \div A) \equiv \varrho_S(A)$. Since $B-S=(A+B)_B$ for $A:B=S$, by (2.9) we have

$$\varrho_S(A) = \gamma_B(A+B)-B. \tag{2.11}$$

Remark that if $A \in \mathscr{M}(S)$, then $A:B=S$ for some $B \in \mathscr{B}_+(\mathscr{H})$, hence by Theorem 2.2

$$\varrho_S(A) = S+B = \lim_{n\to\infty} A:nB.$$

On the other hand

$$A:nB = \frac{n}{n-1}\{A:(n-1)A:(n-1)B\} = \frac{n}{n-1}\{A:(n-1)S\},$$

and letting $n\to\infty$ we have

$$\varrho_S(A) = \lim_{n\to\infty} A:nS.$$

An operator $A \in \mathscr{M}(S)$ is called ϱ_S-*closed* if $\varrho_S(A)=A$; the set of all ϱ_S-closed operators will be denoted by $\mathscr{M}[S]$. For an operator $A \in \mathscr{M}(S)$ the following three conditions are equivalent [11];

$$A \in \mathscr{M}[S] \Leftrightarrow A-S \in [\mathscr{H}]_S \Leftrightarrow \overline{A^{-1/2}\mathscr{R}(S^{1/2})} = \mathscr{H}. \tag{2.12}$$

Notice further that the following conditions are equivalent;

$$A \in \mathscr{M}[S] \Leftrightarrow A+B \in [\mathscr{H}]_B \Leftrightarrow A+B \in [\mathscr{H}]_{B^{1/2}}. \tag{2.13}$$

Now let us cite an assertion, proved in [12].

Theorem 2.3. *Given $A \in \mathscr{B}_+(\mathscr{H})$, let $A_{\mathscr{L}}$ be its short to an operator range $\mathscr{L} \subset \mathscr{H}$. Then $A_{\mathscr{L}}=(A:L)\div L$ for any $L \in \mathscr{B}_+(\mathscr{H})$ such that $\mathscr{R}(L^{1/2})=\mathscr{L}$.*

Applying to this representation of a short the expression (2.10) for parallel subtraction and (2.3) for parallel addition, we obtain

Corollary. *Under the assumptions of the theorem*

$$(A_{\mathscr{L}}f, f) = \sup_g \inf_h \{(Lh, h)-(Lg, g)+(A(f+g+h), f+g+h)\} \quad (\forall f \in \mathscr{H}).$$

Comparison of Theorems 2.2 and 2.3 leads to the identity, first proved in [7]

$$A_{\mathscr{L}} = \lim_{n\to\infty} A:nL \quad (\mathscr{L} = \mathscr{R}(L^{1/2})). \tag{2.14}$$

If $S=A:L$, then clearly

$$A_{\mathscr{L}} = \varrho_S(A) = \lim_{n\to\infty} A:nS. \tag{2.15}$$

The following properties of short operation to an operator range follows immediately from definition (1.7) and relation (2.14):

$$1.\ \ A_{\mathscr{L}} \leqq A; \qquad 2.\ \ (\alpha A)_{\mathscr{L}} = \alpha A_{\mathscr{L}}; \qquad 3.\ \ (A_{\mathscr{L}})_{\mathscr{L}} = A_{\mathscr{L}};$$

$$4.\ \ (A+B)_{\mathscr{L}} \geqq A_{\mathscr{L}}+B_{\mathscr{L}}; \qquad 5.\ \ (A:B)_{\mathscr{L}} \leqq A_{\mathscr{L}}:B_{\mathscr{L}} \leqq A_{\mathscr{L}}:B,\, A:B_{\mathscr{L}}.$$

In view of [5] if $\overline{\mathscr{L}}=\mathscr{L}$ equality sign occurs in the last part. But if $\overline{\mathscr{L}}\neq\mathscr{L}$, equality sign may break down. In fact, let $\mathscr{L}\neq\overline{\mathscr{L}}=\mathscr{H}$, $A=I$, $B\neq 0$ and $\mathscr{R}(B^{1/2})\cap\mathscr{L}=\{0\}$. Then $(A:B)_{\mathscr{L}}=A_{\mathscr{L}}:B_{\mathscr{L}}=A:B_{\mathscr{L}}=0$, but $A_{\mathscr{L}}:B=B(I+B)^{-1}\neq 0$.

It is known [2] that when $\mathscr{L}$ is a closed subspace then $A_n\downarrow A$ implies $(A_n)_{\mathscr{L}}\downarrow A_{\mathscr{L}}$. This property breaks down when $\mathscr{L}$ is a non-closed operator range. We can only assert the following

$$A_n\downarrow A \Rightarrow A_{\mathscr{L}} \leqq \lim_{n\to\infty} (A_n)_{\mathscr{L}} \leqq A_{\overline{\mathscr{L}}}.$$

In fact, since clearly $(A_n)_{\mathscr{L}}\leqq(A_n)_{\overline{\mathscr{L}}}$, letting $\alpha\to\infty$ in the inequalities

$$A:\alpha L \leqq A_n:\alpha L \leqq (A_n)_{\mathscr{L}} \leqq (A_n)_{\overline{\mathscr{L}}}$$

where $L\in\mathscr{B}_+(\mathscr{H})$ such that $\mathscr{R}(L^{1/2})=\mathscr{L}$, we have

$$A_{\mathscr{L}} \leqq (A_n)_{\mathscr{L}} \leqq (A_n)_{\overline{\mathscr{L}}}.$$

It remains to remark that the sequence $\{(A_n)_{\mathscr{L}}\}$ monotonely decreases and $\lim_{n\to\infty}(A_n)_{\overline{\mathscr{L}}}= =A_{\overline{\mathscr{L}}}$.

If $A_{\mathscr{L}}\neq A_{\overline{\mathscr{L}}}$ (as in the example for Corollary 2 to Theorem 1.4), then putting $A_n=A+\frac{1}{n}I$ $(n=1, 2, \ldots)$ we have

$$A_n\downarrow A, \quad (A_n)_{\mathscr{L}} = \left(A+\frac{1}{n}I\right)_{\mathscr{L}} = \left(A+\frac{1}{n}I\right)_{\overline{\mathscr{L}}} \geqq A_{\overline{\mathscr{L}}}$$

(shorts of an invertible operator to $\mathscr{L}$ and $\overline{\mathscr{L}}$ coincide because of formule (1.8)). Therefore, in the present case $\lim_{n\to\infty}(A_n)_{\mathscr{L}}=A_{\overline{\mathscr{L}}}\neq A_{\mathscr{L}}$.

Remark that if $\mathscr{L}_1\supset\mathscr{L}_2$ then $A_{\mathscr{L}_1}\geqq A_{\mathscr{L}_2}$ and $A_{\mathscr{L}_1\cap\mathscr{L}_2}=(A_{\mathscr{L}_1})_{\mathscr{L}_2}$. It follows easily from this that $\{\lim_{n\to\infty}(A_n)_{\mathscr{L}}\}_{\mathscr{L}}=A_{\mathscr{L}}$.

In view of [5], for any closed subspaces $\mathscr{L}_1$ and $\mathscr{L}_2$ the following identity holds

$$A_{\mathscr{L}_1\cap\mathscr{L}_2} = (A_{\mathscr{L}_1})_{\mathscr{L}_2}.$$

For non-closed $\mathcal{L}_1$, $\mathcal{L}_2$ this may break down. For instance, if $\mathcal{L}_1 \cap \mathcal{L}_2 = \{0\}$, $\overline{\mathcal{L}_1} = \overline{\mathcal{L}_2} = \mathcal{H}$ (examples of such operator ranges can be seen in [15]) and $A = I$, then $A_{\mathcal{L}_1 \cap \mathcal{L}_2} = 0$ but $(A_{\mathcal{L}_1})_{\mathcal{L}_2} = I$.

In the case of arbitrary operator ranges $\mathcal{L}_i = \mathcal{R}(L_i^{1/2})$ $(i = 1, 2)$ it follows from the inequalities

$$A : n(L_1 : L_2) = (A : nL_1) : nL_2 \leqq A_{\mathcal{L}_1} : nL_2 \leqq (A_{\mathcal{L}_1})_{\mathcal{L}_2}$$

that $A_{\mathcal{L}_1 \cap \mathcal{L}_2} \leqq (A_{\mathcal{L}_1})_{\mathcal{L}_2}$. Further we have

$$A_{\mathcal{L}_1 \cap \mathcal{L}_2} \leqq 2(A_{\mathcal{L}_1} : A_{\mathcal{L}_2}). \tag{2.16}$$

In fact, denote by P_1, P_2 and P the orthoprojections to the subspaces $\overline{A^{-1/2}\mathcal{L}_1}$, $\overline{A^{-1/2}\mathcal{L}_2}$ and $\overline{A^{-1/2}(\mathcal{L}_1 \cap \mathcal{L}_2)}$ respectively. Since $2(P_1 : P_2)$ becomes the orthoprojection to the intersection of the subspaces $\overline{A^{-1/2}\mathcal{L}_1}$ and $\overline{A^{-1/2}\mathcal{L}_2}$ and $\overline{A^{-1/2}(\mathcal{L}_1 \cap \mathcal{L}_2)} \subset \overline{A^{-1/2}\mathcal{L}_1} \cap \overline{A^{-1/2}\mathcal{L}_2}$, we have

$$P \leqq 2(P_1 : P_2).$$

Multiplying both sides of this inequality by $A^{1/2}$ from left and right, we have inequality (2.16) by Theorem 1.4 and Lemma 2.1. Here in (2.16) equality sign can occur exactly when $\overline{A^{-1/2}(\mathcal{L}_1 \cap \mathcal{L}_2)} = \overline{A^{-1/2}\mathcal{L}_1} \cap \overline{A^{-1/2}\mathcal{L}_2}$. In particular, it is the case when $\mathcal{L}_1$ and $\mathcal{L}_2$ are closed.

On the other hand, it is easy to construct an example in which one of operator ranges $\mathcal{L}_1$ and $\mathcal{L}_2$ is non-closed and $A_{\mathcal{L}_1 \cap \mathcal{L}_2} \neq 2(A_{\mathcal{L}_1} : A_{\mathcal{L}_2})$. In fact, if $A \in \mathcal{B}_+(\mathcal{H})$ and $\mathcal{R}(A) \neq \overline{\mathcal{R}(A)} = \mathcal{H}$, then with a dense operator range $\mathcal{M}_1 \neq \mathcal{H}$ and any operator range $\mathcal{M}_1 \neq \{0\}$ such that $\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$, let $\mathcal{L}_i = A^{1/2}\mathcal{M}_i$ $(i = 1, 2)$. Then clearly $P = 0$ but $2(P_1 : P_2)$ is the orthoprojection to $\overline{\mathcal{M}_2} \neq \{0\}$.

In concluding this section let us show that if an operator range $\mathcal{L}$ is non-closed then the sets (0.1) and (1.10) cannot posses maximum elements. To this end, let us consider the example, used earlier in [11]].

Let A be an operator on $\mathcal{H}$, $0 \leqq A \leqq I$, for which 1 is a continuous spectrum, and $B = I - A$. Then with $S = A : B = A(I - A)$, by (2.13) $A \in \mathcal{M}[S]$. With $\mathcal{L} = \mathcal{R}(B^{1/2})$, by (2.12) we see that $A = A_{\mathcal{L}} = \lim_{n \to \infty} A : nB$. According to the definition of the sets (0.1) and (1.10) we have for all $n = 1, 2, \ldots$

$$A : nB \in \mathcal{X}(A, \mathcal{L}) \cap \mathcal{X}'(A, \mathcal{L}),$$

so that $\mathcal{R}(A : nB) \subset \mathcal{R}((A : nB)^{1/2}) \subset \mathcal{L}$. Therefore if C (resp. C') is the maximum operator in $\mathcal{X}(A, \mathcal{L})$ (resp. $\mathcal{X}'(A, \mathcal{L})$), then $A = \lim_{n \to \infty} A : nB \leqq C$ (resp. C'), hence $A = C$ (resp. C'). But the equality $A = C$ implies that $\mathcal{R}(A) \subset \mathcal{R}(B^{1/2}) \subset \mathcal{R}(S^{1/2})$, that is, for some $\lambda > 0$, $A^2 \leqq \lambda A(I - A)$. Thus $I - A \geqq (1 + \lambda)^{-1} I$, which contradicts that 1 is a continuous spectrum of A. Impossibility of the equality $A = C'$ is proved in an analogous way.

3. Short and extreme points

In the theory of characteristic operator functions, on studying completely non-regular factorizations the description of solutions of the following equation plays a basic role;

(3.1) $$(A-X):(B+X) = 0 \quad (A, B\in\mathscr{B}_+(\mathscr{H}), 0 \leqq X \leqq A).$$

Formulas for this problem were announced in [14, 12]. In this section, together with the proofs of those formulas, we shall present, as a consequence, some propositions, related to extreme point of some convex sets of non-negative operators in a Hilbert space. In particular, short to an operator range is interpreted as an extreme point.

In analogy to the scalar case an *operator interval* $[B, C]$ $(0\leqq B\leqq C)$ will mean the set

$$[B, C] = \{X\in\mathscr{B}_+(\mathscr{H});\ B \leqq X \leqq C\}.$$

The class of all solutions of equation (3.1) will be denoted by $\mathscr{E}(A, B)(\subset[0\ \ A])$.

Theorem 3.1. *Let* $A, B\in\mathscr{B}_+(\mathscr{H})$ *and* $S=A:B$. *Then*

1) *an operator* $X\in[0, A]$ *becomes a solution of equation* (3.1) *exactly when it admits a representation*

(3.2) $$X = (A+B)^{1/2}P(A+B)^{1/2}-B,$$

where P is an orthoprojection to a subspace of $\overline{\mathscr{R}(A+B)}$;

2) *the operator* $X_0=\varrho_S(A)$ *is a minimum solution of equation* (3.1).

Proof. 1) In fact, if $X\in[0, A]$, by definition of parallel sum (2.2)

$$(A-X):(B+X) = B+X-(A+B)_{B+X}.$$

Therefore if $X\in\mathscr{E}(A, B)$, then

(3.3) $$B+X = \omega\omega^*,$$

where according to (1.1) an operator ω is uniquely defined by the conditions $B+X=$ $=\omega(A+B)^{1/2}$, $\mathscr{N}(\omega)\supset\mathscr{N}(A+B)$.

Let $\omega^*=W(\omega\omega^*)^{1/2}$ be the polar representation of ω^*; here W is a partial isometry with initial space $\overline{\mathscr{R}(\omega)}$ and final space $\mathscr{R}(W)=\overline{\mathscr{R}(\omega^*)}(\subset\overline{\mathscr{R}(A+B)})$. It follows from the equality $\omega\omega^*=\omega(A+B)^{1/2}$ that

$$\omega\omega^* = (\omega\omega^*)^{1/2}W^*(A+B)^{1/2},$$

and since $\mathscr{R}(\omega^*)\subset\overline{\mathscr{R}(\omega)}$, $(\omega\omega^*)^{1/2}=W^*(A+B)^{1/2}$. Hence $\omega^*=W(\omega\omega^*)^{1/2}=$

$=WW^*(A+B)^{1/2}$, and consequently

$$X = \omega\omega^* - B = (A+B)^{1/2} P (A+B)^{1/2} - B,$$

where P is the orthoprojection to $\mathscr{R}(W)$.

Conversely, if $X \in \mathscr{B}_+(\mathscr{H})$ is represented in the form (3.2), then clearly $X \leqq A$, and since

$$B + X = (A+B)^{1/2} P (A+B)^{1/2}, \quad \mathscr{N}((A+B)^{1/2} P) \subset \mathscr{N}(A+B),$$

by (1.1) we have $(A+B)_{B+X} = B+X$. Thus $(A-X):(B+X)=0$.

2) It remains to prove that $X_0 = \varrho_S(A)$ is a minimum solution of equation (3.1). For this, remark first that if $X \in \mathscr{E}(A, B)$ then by (3.3) $B \leqq \omega\omega^*$, where, as clarified in the first part of the proof, $\omega = (A+B)^{1/2} P$. Then by [20] there is a contraction $V \in \mathscr{B}(\mathscr{H})$, for which

$$B^{1/2} = V\omega^* = VP(A+B)^{1/2}.$$

Since here $\mathscr{N}(B^{1/2} VP) \supset \mathscr{N}((A+B)^{1/2})$, we have

$$(A+B)_B = B^{1/2} VPV^* B^{1/2}. \tag{3.4}$$

On the other hand, by (3.3) again

$$B = B^{1/2} V\omega^* = B^{1/2} VW(B+X)^{1/2},$$

and since $\mathscr{N}(W) = \mathscr{N}(\omega\omega^*) = \mathscr{N}(B+X)$, we have

$$(X+B)_B = B^{1/2} VWW^* V^* B^{1/2} = B^{1/2} VPV^* B^{1/2}. \tag{3.5}$$

Now we obtain from (3.4) and (3.5)

$$X:B = B - (B+X)_B = B - (A+B)_B = A:B.$$

This implies that if $X \in \mathscr{E}(A, B)$ then $X \leqq \varrho_S(A)$. But by (2.1) and (1.3) we have

$$\varrho_S(A) = \gamma_B(A+B) - B = (A+B)^{1/2} P_{\mathscr{M}} (A+B)^{1/2} - B,$$

where $P_{\mathscr{M}}$ is the orthoprojection to the subspace $\mathscr{M} = \overline{(A+B)^{-1/2}\mathscr{R}(B)} \cap \overline{\mathscr{R}(A+B)}$. Thus $\varrho_S(A)$ admits a representation of the form (3.2), hence by 1) $\varrho_S(A) \in \mathscr{E}(A, B)$. Thus $\varrho_S(A) = \min \mathscr{E}(A, B)$, which is to prove.

Corollary 1. *If* $X = \alpha X_1 + (1-\alpha) X_2 \in \mathscr{E}(A, B)$, *where* $X_1, X_2 \in \mathscr{B}_+(\mathscr{H})$, $0 < \alpha < 1$, *then* $X_1 = X_2 \in \mathscr{E}(A, B)$.

If fact, according to inequality (2.6)

$$(A-X):(B+X) = \{\alpha(A-X_1) + (1-\alpha)(A-X_2)\} : \{\alpha(B+X_1) + (1-\alpha)(B+X_2)\} \geqq$$
$$\geqq \alpha(A-X_1):(B+X_1) + (1-\alpha)(A-X_2):(B+X_2).$$

Therefore if $X \in \mathscr{E}(A, B)$, then $X_1, X_2 \in \mathscr{E}(A, B)$. Denoting by P, P_1 and P_2 the orthoprojections, corresponding to X, X_1 and X_2 in (3.2) respectively, we have

$$P = \alpha P_1 + (1-\alpha) P_2,$$

clearly this last equality is possible only when $P=P_1=P_2$, hence $X=X_1=X_2$.

Corollary 2. *If $\mathscr{P}_A$ denotes the class of all orthoprojections to subspaces of $\overline{\mathscr{R}(A)}$, then*

$$\mathscr{E}(A, 0) = \{X\in[0, A]\colon X = A^{1/2}PA^{1/2},\ P\in\mathscr{P}_A\}. \tag{3.6}$$

Recall that X is called an *extreme point* of a convex set $\mathscr{S}$, if the relation

$$X = \alpha X_1 + (1-\alpha) X_2$$

where $X_1, X_2 \in \gamma$, $0<\alpha<1$, is possible only when $X_1=X_2=X$. The class of all extreme points a set $\mathscr{S}$ is denoted by $\operatorname{ex}\mathscr{S}$.

Corollary 3. $\mathscr{E}(A, 0) = \operatorname{ex}[0, A]$.

In fact, if $X\in\mathscr{E}(A, 0)$ and $X=\alpha X_1+(1-\alpha)X_2$, where $X_1, X_2\in[0, A]$, $0<\alpha<1$, then by Corollary 1 $X_1=X_2=X$, that is, $X\in\operatorname{ex}[0, A]$. If $X\notin\mathscr{E}(A, 0)$, then $X\neq X_1\equiv \equiv X+(A-X)\colon X\in[0, A]$, $X\neq X_2\equiv X-(A-X)\colon X\in[0, A]$ and $X=\frac{1}{2}(X_1+X_2)$, so that $X\notin\operatorname{ex}[0, A]$.

Remark. The relation $\mathscr{E}(A, 0)=\operatorname{ex}[0, A]$ is found in [13], where it is proved in essence that if $X_1, X_2\in\operatorname{ex}[0, A]$ then $2(X_1\colon X_2)\in\operatorname{ex}[0, A]$. This follows also from Corollary 2 by Lemma 2.1, because

$$2(X_1\colon X_2) = 2\big((A^{1/2}P_1A^{1/2})\colon(A^{1/2}P_2A^{1/2})\big) = A^{1/2}\big(2(P_1\colon P_2)\big)A^{1/2} = A^{1/2}P_{\mathscr{M}}A^{1/2},$$

where P_1 and P_2 are the orthoprojections corresponding to X_1 and X_2 through (3.6), while $P_{\mathscr{M}}$ is the orthoprojection to the subspace $\mathscr{M}=\mathscr{R}(P_1)\cap\mathscr{R}(P_2)$.

Let $A, B\in\mathscr{B}_+(\mathscr{H})$ and $C=A+B$. Remark that $X\in\operatorname{ex}[B, C]$ exactly then $X-B\in\operatorname{ex}[0, A]$ thus by Corollaries 2 and 3 to Theorem 3.1

$$\operatorname{ex}[0, A] = \{X\in\mathscr{B}_+(\mathscr{H})\colon X = A^{1/2}PA^{1/2},\ P\in\mathscr{P}_A\} \tag{3.7}$$

hence

$$\operatorname{ex}[B, C] = \{X\in\mathscr{B}_+(\mathscr{H})\colon X = A^{1/2}PA^{1/2}+B,\ P\in\mathscr{P}_A\}. \tag{3.8}$$

Obviously $\operatorname{ex}[B, C]$ contains all points of $\operatorname{ex}[0, C]$ in $[B, C]$. The converse is not true, if $B\notin\operatorname{ex}[0, C]$. Relation (3.8) implies that in this case $\operatorname{ex}[B, C]$ contains, together with B, other points not belonging to $\operatorname{ex}[0, C]$.

Indeed, in the contrary case, taking a sequence $\{P_n\}\subset\mathscr{P}_A$ such that $0\neq P_n\to 0$ $(n\to\infty)$, we have a sequence $\{Q_n\}\subset\mathscr{P}_C$ for which

$$A^{1/2}P_nA^{1/2}+B = C^{1/2}Q_nC^{1/2} \quad (n = 1, 2, \ldots).$$

Since there exists a limit $Q=\lim_{n\to\infty} Q_n$, and $B=C^{1/2}QC^{1/2}$ $(Q\in\mathscr{P}_C)$, hence $B\in\operatorname{ex}[0, C]$, contradicting the assumption.

Theorem 3.2. *Let* $B\in[0, C]$ $\big(C\in\mathscr{B}_+(\mathscr{H})\big)$. *Then* $\operatorname{ex}[B, C]\subset\operatorname{ex}[0, C]$ *exactly when* $B\in\operatorname{ex}[0, C]$.

Proof. Since always $B\in\operatorname{ex}[B, C]$, it suffices to consider the case $B\in\operatorname{ex}[0, C]$. With $A=C-B$, remark that since $\operatorname{ex}[0, C]=\mathscr{E}(C, 0)$ we have $A:B=(C-B):B=0$. Thus by (2.8)

$$\mathscr{R}(A^{1/2})\cap\mathscr{R}(B^{1/2}) = \mathscr{R}\big((A:B)^{1/2}\big) = \{0\}.$$

Then, with $\mathscr{L}=\mathscr{R}(B^{1/2})$, for any $P\in\mathscr{P}_A$

$$\mathscr{R}\big((A-A^{1/2}PA^{1/2})^{1/2}\big)\cap\mathscr{R}\big((A^{1/2}PA^{1/2}+B)^{1/2}\big) =$$

$$= \mathscr{R}\big(A^{1/2}(1-P)\big)\cap\big(\mathscr{R}(A^{1/2}P)+\mathscr{L}\big) = \{0\},$$

hence

$$(A-A^{1/2}PA^{1/2}):(B+A^{1/2}PA^{1/2}) = 0.$$

Since this relation means that each operator $X=A^{1/2}PA^{1/2}+B$ is contained in $\mathscr{E}(C, 0)(P\in\mathscr{P}_A)$. Therefore in view of (3.8) we can conclude that $\operatorname{ex}[B, C]\subset\operatorname{ex}[0, C]$, what is to prove.

Remark. Let $A, B\in\mathscr{B}_+(\mathscr{H})$, and $C=A+B$. Then $B\in\operatorname{ex}[0, C]$ if and only if $A\in\operatorname{ex}[0, C]$. In this case it follows from the already proved fact and relation (3.8) that $\operatorname{ex}[0, A]\subset\operatorname{ex}[0, C]$. In other words, if $A=C^{1/2}PC^{1/2}$ $(P\in\mathscr{P}_C)$ and $X=A^{1/2}XA^{1/2}=A^{1/2}QA^{1/2}$ $(Q\in\mathscr{P}_A)$, then $X=C^{1/2}RC^{1/2}$ for some $R\in\mathscr{P}_C$.

Theorem 3.3. *Suppose that there are given an operator* $A\in\mathscr{B}_+(\mathscr{H})$ *and an operator range* $\mathscr{L}\subset\mathscr{H}$. *If* $\mathscr{L}=\mathscr{R}(B^{1/2})$ $\big(B\in\mathscr{B}_+(\mathscr{H})\big)$, *then*

$$A_{\mathscr{L}} = \min\{\operatorname{ex}[0, A+B]\cap\operatorname{ex}[B, A+B]\}-B.$$

Proof. In view of (3.7) and (3.8)

$$X\in\operatorname{ex}[0, A+B]\cap\operatorname{ex}[B, A+B]$$

if and only if there are representations

$$X = (A+B)^{1/2}P(A+B)^{1/2} = A^{1/2}P'A^{1/2}+B,$$

where $P\in\mathscr{P}_{A+B}$ and $P'\in\mathscr{P}_A$. Here P, which runs over all admissible elements of $\mathscr{P}_{A+B}$, can attain a minimum value for the operator

$$X-B = (A+B)^{1/2}P(A+B)^{1/2}-B.$$

In view of Theorem 3.1 this minimum value exists and coincides with $\varrho_S(A)$ $(S=A:B)$

$$X_{min}-B = \varrho_S(A).$$

On the other hand, by (2.15) $A_{\mathscr{L}}=\varrho_S(A)$. This proves the theorem.

It is easy to see that if $A\in\mathscr{B}_+(\mathscr{H})$ and $\mathscr{L}=\mathscr{R}(B^{1/2})=\mathscr{R}(B_1^{1/2})$, where $B, B_1\in\mathscr{B}_+(\mathscr{H})$, then equations $(A-X):(B+X)=0$ and $(A-X):(B_1+X)=0$ are equivalent. Thus their solutions, determined by A and $\mathscr{L}$, will be denoted by $\mathscr{E}(A, \mathscr{L})\big(=\mathscr{E}(A, B)=\mathscr{E}(A, B_1)\big)$.

Theorem 3.4. *For an operator $A\in\mathscr{B}_+(\mathscr{H})$ and an operator range $\mathscr{L}\subset\mathscr{H}$ the following relation holds*

$$\mathscr{E}(A, \mathscr{L}) = \operatorname{ex}[A_{\mathscr{L}}, A].$$

Proof. Indeed, if $X\in\mathscr{E}(A, \mathscr{L})$ and $\mathscr{L}=\mathscr{R}(B^{1/2})$ $\big(B\in\mathscr{B}_+(\mathscr{H})\big)$, then

$$(A-X):X \leqq (A-x):(B+X) = 0,$$

hence $X\in\operatorname{ex}[0, A]$. But by Theorem 3.1 we have $X\in[A_{\mathscr{L}}, A]$, hence $X\in\operatorname{ex}[A_{\mathscr{L}}, A]$.

Suppose, conversely, that $X\in\operatorname{ex}[A_{\mathscr{L}}, A]$. Since by (1.8) $A_{\mathscr{L}}=A^{1/2}P_{\mathscr{M}}A^{1/2}$, where $P_{\mathscr{M}}$ is the orthoprojection to the subspace $\mathscr{M}=\overline{A^{-1/2}\mathscr{L}}$, we have $A_{\mathscr{L}}\in\operatorname{ex}[0, A]$ and by Theorem 3.2 $\operatorname{ex}[A_{\mathscr{L}}, A]\subset\operatorname{ex}[0, A]$. Thus $X\in\operatorname{ex}[0, A]$, that is, $(A-X):X=0$. But by (2.8) this relation means that

$$\mathscr{R}\big((A-X)^{1/2}\big)\cap\mathscr{R}(X^{1/2}) = \{0\}. \tag{3.9}$$

Further, it follows from the relations

$$A:B \leqq A_{\mathscr{L}} \leqq X \leqq A\big(B\in\mathscr{B}_+(\mathscr{H}),\ \mathscr{R}(B^{1/2}) = \mathscr{L}\big)$$

that

$$\mathscr{R}(A^{1/2})\cap\mathscr{R}(B^{1/2}) \subset \mathscr{R}(A_{\mathscr{L}}^{1/2}) \subset \mathscr{R}(X^{1/2}) \subset \mathscr{R}(A^{1/2}).$$

Now it is clear that $(A-X):(B+X)=0$, since

$$\mathscr{R}\big((A-X)^{1/2}\big)\cap\mathscr{R}\big((B+X)^{1/2}\big) = \{0\}.$$

In fact, if for some $f, g, h\in\mathscr{H}$

$$(A-X)^{1/2}f = B^{1/2}g+X^{1/2}h,$$

then $B^{1/2}g\in\mathscr{R}(A^{1/2})\cap\mathscr{R}(B^{1/2})\subset\mathscr{R}(X^{1/2})$ and by (3.9) $(A-X)^{1/2}f=0$. This completes the proof.

Remark. If $X\in[0, A]$, then $\mathscr{R}(X)\subset\mathscr{R}(X^{1/2})\subset\mathscr{R}(A^{1/2})$, that is, $A\in\{\mathscr{H}\}_X$ and by definition of parallel sum (2.2) $(A-X):X=X-A_X$. Consequently, $X\in\operatorname{ex}[0, A]$ exactly when $X=A_X$. For invertible A this condition $(X=XA^{-1}X)$ is found in

[13], in which there is proved that

$$X \in \mathrm{ex}\,[0, A] \Leftrightarrow D_A X = X,$$

where by definition

$$D_A X = \sup_{\lambda > 0} \frac{X - Q_\lambda^A X}{\lambda} \quad \text{and} \quad Q_\lambda^A X = \sup_\lambda X : \frac{1}{\lambda} A.$$

Remark, in this connection, that there follows from Proposition 6.2 of [13] and formula (1.2) the relation $D_A X = A_X$, valid for all X in the domain of definition of D_A, namely under the condition $A \in \{\mathscr{H}\}_X$.

References

[1] M. G. Krein, Theory of selfadjoint extensions of semi-bounded Hermitian operators, *Mat. Sb.*, **10** (1947), 431—495 (in Russian).

[2] U. L. Smul'jan, Operator Hellinger integral, *Mat. Sb.*, **49** (1959), 381—430 (in Russian).

[3] U. L. Smul'jan, Some problems of theory of operators with finite non-Hermitian rank, *Mat. Sb.*, **57** (1962), 105—136 (in Russian).

[4] E. L. Pekarev, Regular and completely non-regular systems of nodes and their characteristic functions, *Izv. Vuz. Mat.*, (1980) no 2, 73—75 (in Russian). [A complete manuscript was deposited in *VINITI*, 1978, no 2677 78 (79).]

[5] W. N. Jr. Anderson and G. E. Trapp, Shorted operators. II, *SIAM J. Appl. Math.*, **28** (1975), 60—71.

[6] K. Nishio and T. Ando, Characterizations of operations derived from network connection, *J. Math. Anal. Appl.*, **53** (1976), 539—549.

[7] T. Ando, Lebesgue-type decomposition of positive operators, *Acta Sci. Math.*, **38** (1976), 253—260.

[8] H. Kosaki, Remarks on Lebesgue-type decomposition of positive operators, *J. Operator Theory*, **11** (1984), 137—143.

[9] T. Ando and W. Szymanski, Order structure and Lebesgue decomposition for positive definite operator functions, *Indiana Univ. Math. J.*, **35** (1986), 157—173.

[10] K. Nishio, Characterization of Lebesgue-type decomposition of positive operators, *Acta Sci. Math.*, **42** (1980), 143—152.

[11] E. L. Pekarev and U. L. Smul'jan, Parallel addition and parallel subtraction of operators, *Izv. AN SSSR*, **40** (1976), 366—387. English translation: *Math. USSR, Izvestija.*

[12] E. L. Pekarev, Short to operator range, *Funkc. Anal. Pril.*, **12** (1978), no 3, 84—85. English translation: *Functional Anal. Appl.*

[13] S. L. Eriksson and H. Leutwiler, A potential theoretic approach to parallel addition, *Math. Ann.*, **274** (1986), 301—317.

[14] E. L. Pekarev, Regular and completely non-regular factorizations of a characteristic function, *Funkc. Anal. Pril.*, **10** (1976), no 1, 85—86. English translation: *Functional Anal. Appl.*

[15] P. A. Fillmore and J. P. Williams, On operator ranges, *Adv. Math.*, **7** (1971), 254—281.

[16] W. N. Anderson and R. J. Duffin, Series and parallel addition of matrices, *J. Math. Anal. Appl.*, **26** (1969), 576—594.

[17] R. J. DUFFIN and T. D. MORLEY, Almost definite operators and electromechanical systems, *SIAM J. Appl. Math.*, **30** (1978), 21—30.

[18] G. B. PAASTY, The parallel sum of non-linear monotone operators, *Nonlinear Analysis*, **10** (1986), 215—227.

[19] R. M. TORREJON, Perturbation of nonlinear operators and their parallel sum, *Nonlinear Analisis*, **10** (1986), 483—490.

[20] R. G. DOUGLAS, On majorization, factorization and range inclusion of operators in Hilbert space, *Proc. Amer. Math. Soc.*, **17** (1966), 413—415.

[21] E. L. PEKAREV, Regularization of a pair of characteristic operator functions. [Deposited in *UkrNIINTI*, 1987, no. 2682—Uk87 (in Russian).]

ODESSA TECHNOLOGICAL INSTITUTE
OF FOOD INDUSTRY
NAMED AFTER M. V. LOMONOSOV
112 SVERDLOV STR., ODESSA-39.
270039, USSR